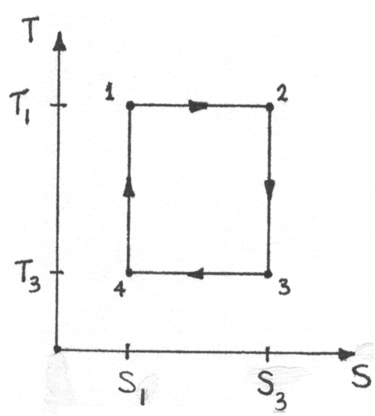
University of California, Berkeley
Physics H7B Spring 1999 (*Strovink*)

SOLUTION TO EXAMINATION 1

Directions. Do all four problems (weights are indicated). This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he or she will not give hints and will be obliged to write your question and its answer on the board. Don't use a calculator, which you don't need – roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (25 points) A heat engine for which the working material is an ideal monatomic gas moves slowly enough that all parts of it are always in mutual equilibrium. It is described by a rectangular path on the T (absolute temperature) – S (entropy) plane, as in the figure. While on the path $1 \to 2$, the gas in the engine takes heat from a bath at high temperature T_1; on the path $3 \to 4$, it returns heat to bath at lower temperature T_3. On the paths $2 \to 3$ and $4 \to 1$, the entropy has constant values S_3 and S_1, respectively.



a. (5 points) Write down the net change

$$(\Delta U_{23} + \Delta U_{41})$$

in internal energy for the sum of the two paths $2 \to 3$ and $4 \to 1$.

b. (5 points) Compute the net change

$$\oint_{12341} T\,dS$$

over one complete cycle of the engine.

c. (8 points) Deduce the value of the mechanical work

$$\oint_{12341} p\,dV$$

done **by** the gas **on** the rest of the universe over one complete cycle of the engine.

d. (7 points) In one cycle, what fraction of the heat withdrawn from the hot reservoir is converted to mechanical work done by the gas on the rest of the universe?

Hint: Keep in mind that the only parameters given in this problem are T_1, T_3, S_1, and S_3; your answers, if nontrivial, must be expressed in terms of these parameters.

Solution:
(a.)
U of an ideal gas is a function only of T, so the isothermal segments $1 \to 2$ and $3 \to 4$ cause no change in U. Therefore

$$(\Delta U_{23} + \Delta U_{41}) = \oint_{12341} dU = 0$$

because U is a state function.
(b.)

$$\oint_{12341} T\,dS = (T_1 - T_3)(S_3 - S_1) ,$$

the area of the rectangle in the figure.

(c.)

$$\begin{aligned}\oint_{12341} p\,dV &= -\oint_{12341} \delta W \\ &= -\oint_{12341} dU + \oint_{12341} \delta Q \\ &= 0 + \oint_{12341} T\,dS \\ &= (T_1 - T_3)(S_3 - S_1)\ .\end{aligned}$$

(d.)

$$\begin{aligned}\frac{\oint_{12341} p\,dV}{Q_2} &= \frac{\oint_{12341} p\,dV}{\int_1^2 T\,dS} \\ &= \frac{(T_1 - T_3)(S_3 - S_1)}{T_1(S_3 - S_1)} \\ &= 1 - \frac{T_3}{T_1}\ .\end{aligned}$$

It is also acceptable to state that this is a Carnot engine and quote this standard result for its efficiency.

2. (25 points) In a hypothetical one-dimensional system, thermal motion of atoms in the y and z directions is "frozen out", so, effectively, the atoms are able to move only in the x direction. In that direction, an atom has velocity v ($-\infty < v < \infty$). The fraction dF of atoms with velocity between v and $v + dv$ is

$$dF \equiv f_v(v)\,dv = \frac{\exp\left(-\frac{mv^2}{2kT}\right)dv}{\int_{-\infty}^{\infty} \exp\left(-\frac{mv^2}{2kT}\right)dv}\ ,$$

where $f_v(v)$ is the probability density (RHK: "relative probability") of the value v, m is the atomic mass, k is Boltzmann's constant, and T is the absolute temperature.

a. (10 points) Calculate the mean value of the square of v, *i.e.* $\langle v^2\rangle$. If you wish, you may leave your answer in the form of a ratio of definite integrals. *Do not merely guess the answer.*

b. (15 points) Define E to be the kinetic energy $\frac{1}{2}mv^2$ of an atom. The fraction dF of atoms with kinetic energy between E and $E+dE$ is

$$dF \equiv f_E(E)\,dE\ ,$$

where $f_E(E)$ is the probability density of the value E. One might imagine $f_E(E)$ to take the possible forms:

$$\begin{aligned}f_E(E) &\propto E^{-1/2}\exp\left(-\frac{E}{kT}\right)\ ? \\ &\propto \exp\left(-\frac{E}{kT}\right)\ ? \\ &\propto E^{1/2}\exp\left(-\frac{E}{kT}\right)\ ? \\ &\propto E\exp\left(-\frac{E}{kT}\right)\ ?\end{aligned}$$

Which one form is correct, and why?

Solution:
(a.)
From the definition of it that is given, this probability density is explicitly normalized:

$$\int_{-\infty}^{\infty} f_v(v)\,dv \equiv 1\ .$$

Using the standard method for taking the average, when f_v is normalized,

$$\begin{aligned}\langle v^2\rangle &= \int_{-\infty}^{\infty} v^2 f_v\,dv \\ &= \frac{\int_{-\infty}^{\infty} v^2 \exp\left(-\frac{mv^2}{2kT}\right)dv}{\int_{-\infty}^{\infty} \exp\left(-\frac{mv^2}{2kT}\right)dv}\ .\end{aligned}$$

This answer is enough to earn full credit. For completeness, defining $\beta \equiv 1/kT$ and $u \equiv \frac{1}{2}mv^2$, we can rewrite this quotient as

$$\begin{aligned}\langle v^2\rangle &= \frac{\frac{2}{m}\int_{-\infty}^{\infty} u^{1/2}\exp(-\beta u)\,du}{\int_{-\infty}^{\infty} u^{-1/2}\exp(-\beta u)\,du} \\ &= -\frac{2}{m}\frac{\partial}{\partial\beta}\ln\left(\int_{-\infty}^{\infty} u^{-1/2}\exp(-\beta u)\,du\right) \\ &= -\frac{2}{m}\frac{\partial}{\partial\beta}\ln\left(C\beta^{-1/2}\right) \\ &= \frac{1}{m\beta} \\ &= \frac{kT}{m}\ ,\end{aligned}$$

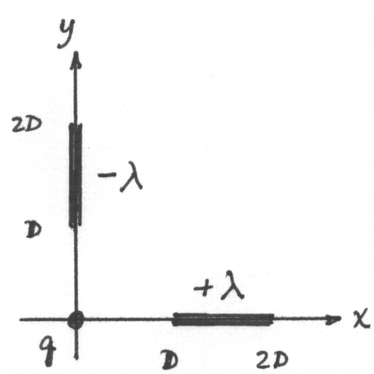
where, in the above, C is a constant whose value is immaterial here.

(b.)

$$\begin{aligned} dF &\equiv f_E(E)\,dE \\ f_E &= \frac{dF}{dE} \\ &= \frac{dF}{dv}\frac{dv}{dE} \\ &\equiv f_v(v)\frac{dv}{dE} \\ &= \frac{f_v(v)}{\frac{d}{dv}\left(\frac{1}{2}mv^2\right)} \\ &= \frac{f_v(v)}{mv} \\ &\propto \frac{\exp\left(-\frac{E}{kT}\right)}{E^{1/2}} \ . \end{aligned}$$

3. (25 points)

A fixed line charge of $+\lambda$ esu/cm on the x axis extends from $x = D$ to $x = 2D$, and a fixed line charge of $-\lambda$ esu/cm on the y axis extends from $y = D$ to $y = 2D$.



a. (10 points) Find the work required to bring a test point charge q from infinity to the origin. Does your answer depend on the path you chose? If so, specify the path.

b. (15 points) Calculate the mechanical force (magnitude and direction) that is required to keep the test charge at the origin.

Solution:

(a.)

For every positive charge element that is a certain distance from the origin, there is a corresponding negative charge element located at the same distance from the origin (but in an orthogonal direction). Therefore, by symmetry, the electrostatic potential ϕ vanishes at the origin, as does the work W required to bring the charge in from infinity:

$$W = q\big(\phi(0) - \phi(\infty)\big) = 0 \ .$$

(b.)

From the positive part of the charge distribution, the electric field at the origin is

$$\begin{aligned} \mathbf{E}_+ &= -\hat{\mathbf{x}} \int_D^{2D} \frac{\lambda}{x^2}\,dx \\ &= -\hat{\mathbf{x}}\Big(\frac{\lambda}{D} - \frac{\lambda}{2D}\Big) \\ &= -\hat{\mathbf{x}}\frac{\lambda}{2D} \ . \end{aligned}$$

Likewise, from the negative part of the charge distribution,

$$\mathbf{E}_- = +\hat{\mathbf{y}}\frac{\lambda}{2D} \ .$$

The total electric field is

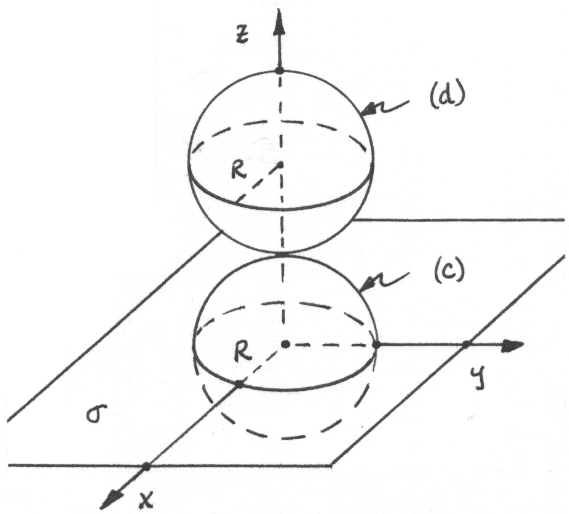
$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\hat{\mathbf{y}} - \hat{\mathbf{x}}}{\sqrt{2}}\frac{\lambda}{D\sqrt{2}} \ .$$

The mechanical force $\mathbf{F}$ required to keep the test charge at the origin must oppose $q\mathbf{E}$:

$$\mathbf{F} = \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}}{\sqrt{2}}\frac{q\lambda}{D\sqrt{2}} \ ,$$

where the first factor is its direction (at $-45°$ to the x axis), and the second is its magnitude.

4. (25 points) The infinite plane $z = 0$ carries a uniform surface charge density σ esu/cm^2. There are no other charges in the problem.



a. (5 points) Find the magnitude and direction of the electric field $\mathbf{E}_+$ everywhere in the region $z > 0$.

b. (5 points) Find the magnitude and direction of the electric field $\mathbf{E}_-$ everywhere in the region $z < 0$.

c. (8 points) Consider a spherical surface of radius R centered at the origin. Find the electric flux

$$\iint \mathbf{E} \cdot d\mathbf{a}$$

through the *top half* (top hemisphere) of this surface.

d. (7 points) Consider a second spherical surface, again of radius R, but now centered at the point $(0, 0, 2R)$, so that it does not enclose any charge. Find the electric flux

$$\iint \mathbf{E} \cdot d\mathbf{a}$$

through the *bottom half* (bottom hemisphere) of this surface.

Solution:
(a.) (b.)
The charge distribution is symmetric about the plane $z = 0$, so

$$\mathbf{E}_+ = -\mathbf{E}_- \ ,$$

and both fields are normal to the $z = 0$ plane. Using a Gaussian pillbox with flat surface area A parallel to the $z = 0$ plane,

$$\begin{aligned}
\oint \mathbf{E} \cdot d\mathbf{a} &= 4\pi Q_{\text{encl}} \\
((E_+)_z - (E_-)_z)A &= 4\pi\sigma A \\
((E_+)_z + (E_+)_z)A &= 4\pi\sigma A \\
(E_+)_z = -(E_-)_z &= 2\pi\sigma \\
\mathbf{E}_+ = -\mathbf{E}_- &= \hat{\mathbf{z}}\, 2\pi\sigma \ .
\end{aligned}$$

It is acceptable simply to recall that the electric field on either side of an infinite sheet of charge has this value, in the absence of other charges.
(c.)
Again because the charge distribution is symmetric about the plane $z = 0$, substituting a sphere of radius R centered at the origin for the Gaussian pillbox used in the solution of part (a.),

$$\begin{aligned}
\iint_{\text{hemi}}^{\text{top}} \mathbf{E} \cdot d\mathbf{a} &= \iint_{\text{hemi}}^{\text{bot}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{2} 4\pi Q_{\text{encl}} \\
\iint_{\text{hemi}}^{\text{top}} \mathbf{E} \cdot d\mathbf{a} &= \frac{1}{2} 4\pi\pi R^2 \sigma \\
&= 2\pi^2 R^2 \sigma \ .
\end{aligned}$$

(d.)
Because $\mathbf{E}$ is constant throughout the semi-infinite region $z > 0$, the flux of $\mathbf{E}$ through the top of the hemisphere centered at $(0, 0, 2R)$ is the same as the flux in part (c.) through the top of the hemisphere centered at the origin. Since the hemisphere centered at $(0, 0, 2R)$ contains no charge, the flux of $\mathbf{E}$ through its bottom half must cancel the flux through its top half. Therefore

$$\iint_{\text{hemi}}^{\text{bot}} \mathbf{E} \cdot d\mathbf{a} = -2\pi^2 R^2 \sigma \ .$$